Notes on Topology

HIRABA, Seiji

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1 Topological spaces=top. sp's

1.1 Topology=top.

$X \neq \emptyset$: a set, 2^X: a family of all subsets of X,

$\mathcal{O} \subset 2^X$: **topology** $\overset{\text{def}}{\iff}$ (1) $X, \emptyset \in \mathcal{O}$. (2) $U, V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$. (3) $U_\lambda \in \mathcal{O}(\lambda \in \Lambda) \Rightarrow \bigcup_{\lambda\in\Lambda} U_\lambda \in \mathcal{O}$.

$U \in \mathcal{O}$ is called an **open set**.

$(X, \mathcal{O})$: a **topological sp.** $\mathcal{O} = \mathcal{O}_X$.

A **closed set** $F \in \mathcal{C} = \mathcal{O}^c \overset{\text{def}}{\iff} F^c \in \mathcal{O}$, (where $\mathcal{O}^c$ is a family of complements of all open subsets, it is not $2^X \setminus \mathcal{O}$. In this paper, a set operation for a family of subsets mean a family of operated elements = sets, e.g. complement, closure, union, intersection, etc.)

$\{X, \emptyset\}$: a **trivial top.** = an **indiscrete top.** In this space, any non-empty sets are connected.

2^X: a **discrete top.** In this space, any subsets are open and closed, and every subsets except singletons are disconnected.

A **relative top. of** $A \subset X$; $\mathcal{O}_A := \mathcal{O}_X \cap A$.

1.2 Metric sp's.

A **metric** $d = d_X : X \times X \to [0, \infty]$: a mapping s.t. ${}^\forall x, y, z \in X$,

(1) $d(x, y) \geq 0, = 0 \iff x = y$ (2) $d(x, y) = d(y, x)$ (3) $d(x, z) \leq d(x, y) + d(y, z)$.

(1) non-negativity, zero-value identity (2) symmetry (3) triangle inequality.

$(X, d) = (X, d_X)$: a **metric sp.**

In $\mathbf{R}^n$, $d(x, y) = |x - y|$ ($|x| = |(x_1, \ldots, x_n)| := (x_1^2 + \cdots + xn^2)^{1/2}$) is a metric, $\mathbf{R}^n$ is called n**-dimensional Euclidean sp.**

A δ**-neighborhood of** x: $U_\delta(x) := \{y \in X; d(x, y) < \delta\}$ ($U_\delta(x) = U(x; \delta) = B(x; \delta) = B_\delta(x)$).

Let $(X, \mathcal{O})$ be a top. sp.

U: an **open set of a metric sp.** $\overset{\text{def}}{\iff}$ ${}^\forall x \in U, {}^\exists \delta > 0; U_\delta(x) \subset U$.

x: a **boundary point=pt of** A $\overset{\text{def}}{\iff}$ ${}^\forall \varepsilon > 0, U_\varepsilon(x) \cap A \neq \emptyset, U_\varepsilon(x) \cap A^c \neq \emptyset$.

x: an **inner pt of** A $\overset{\text{def}}{\iff}$ ${}^\exists \delta > 0; U_\delta(x) \subset A$.

x: an **outer pt of** A $\overset{\text{def}}{\iff}$ ${}^\exists \delta > 0; U_\delta(x) \subset A^c$.

∂A: the **boundary** of A, A^o: the **interior** of A, $(\overline{A})^c$: the **exterior** of A.

$\overline{A} = A \cup \partial A$: the **closure** of A.

C: a **closed set of a metric sp.** $\overset{\text{def}}{\iff}$ $\partial C \subset C \iff C = \overline{C}. \iff C^c$: open.

1.3 A system of neighborhoods, a topological basis = an open basis

Let $(X, \mathcal{O} = \mathcal{O}_X)$ be a top. sp.

${}^\forall x \in X$ fix. $V \subset X$: a **neighborhood=nbd** of x $\overset{\text{def}}{\iff}$ ${}^\exists U \in \mathcal{O}; x \in U \subset V$.

$\mathcal{N}(x)$: a system of nbds=a family of all nbds of x. $\mathcal{N}_{\mathcal{O}}(x) := \mathcal{N}(x) \cap \mathcal{O}$, $\mathcal{N}_{\mathcal{C}}(x) := \mathcal{N}(x) \cap \mathcal{C}$

Clearly, **a system of open nbds is a collection of all closures of open nbds**, i.e., $\mathcal{N}_{\mathcal{C}}(x) = \overline{\mathcal{N}_{\mathcal{O}}(x)} = \{\overline{U}; U \in \mathcal{N}_{\mathcal{O}}(x)\}$.

$U \in \mathcal{N}_{\mathcal{O}}(x) \iff x \in U \in \mathcal{O}$. Hence $U \in \mathcal{O} \iff {}^\forall x \in U, {}^\exists V_x \in \mathcal{N}_{\mathcal{O}}(x); x \in V_x \subset U$

Question Show the above equivalent. $\Rightarrow$ Clear. $\Leftarrow$ $U = \bigcup_{x \in U} V_x$ is open. ■

$\mathcal{B} \subset \mathcal{O}$: a **topological basis** = an **open basis** $\overset{\text{def}}{\iff}$ $\{\bigcup \mathcal{U}; \mathcal{U} \subset \mathcal{B}\} \equiv \{\bigcup_\lambda U_\lambda; \{U_\lambda\} \subset \mathcal{B}\} = \mathcal{O}$.

$\mathcal{B}_0 \subset \mathcal{O}$: a **quasi-basis** $\overset{\text{def}}{\iff}$ $\{\bigcap_{k=1}^n B_k; B_k \in \mathcal{B}_0, k = 1, 2, \ldots, n, n \geq 0\}$ is a top. basis, i.e., $\{\bigcup_\lambda \bigcap_{k=1}^{n_\lambda} B_{\lambda,k}; B_{\lambda,k} \in \mathcal{B}_0, k = 1, 2, \ldots, n_\lambda, n_\lambda \geq 0\} = \mathcal{O}$, where if $n = 0$, then $\bigcap_{k=1}^n B_k = X$.

In $\mathbf{R}$, $\{(-\infty, b), (a, \infty); a, b \in \mathbf{Q}\}$ is a quasi-basis.

In $\mathbf{R}^n$, $\{U_r(q); r \in \mathbf{Q}_+, q \in \mathbf{Q}^n\}$, $\{\prod_{k=1}^n (a_k, b_k); a_k, b_k \in \mathbf{Q}\}$ are top. basises.

By these, $\mathbf{R}^n$ satisfies 2nd axiom of countability.

$\mathcal{N}_0(x) \subset \mathcal{N}(x)$: a **neighborhood basis** $\overset{\text{def}}{\iff}$ ${}^\forall V \in \mathcal{N}(x), {}^\exists U(x) \in \mathcal{N}_0(x); x \in U(x) \subset V$.

In $\mathbf{R}^n$, $\{U_{1/n}(x); n \geq 1\}$ is a nbd basis.

1st axiom of countability $=^\exists$ a countable nbd basis.

2nd axiom of countability $= {}^\exists$ a countable top. basis.

Clearly, 2nd axiom of countability $\Rightarrow$ 1st one.

1.4 Continuous mappings=conti. map's

A map. $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is **continuous=conti.** $\overset{\text{def}}{\iff}$ ${}^\forall V \in \mathcal{O}_Y, f^{-1}(V) \in \mathcal{O}_X$, i.e., $f^{-1}(\mathcal{O}_Y) \subset \mathcal{O}_X$. $\iff f^{-1}(\mathcal{C}_Y) \subset \mathcal{C}_X$. ($x \in f^{-1}(V) \overset{\text{def}}{\iff} f(x) \in V$).

Note that $f : X \to Y; x \mapsto f(x)$: a **mapping=map.** $\overset{\text{def}}{\iff}$ ${}^\forall x \in X, {}^{\exists_1} y \in Y; y = f(x)$.

In a metric sp., a map. $f:(X,d_x)\to(Y,d_Y)$ is conti. $\overset{\text{def}}{\iff}$ ${}^\forall x\in X, f$: conti. at x. $\iff$ ${}^\forall x\in X, {}^\forall\varepsilon>0, {}^\exists\delta>0; {}^\forall x'\in X; d_X(x,x')<\delta, d_Y(f(x),f(x'))<\varepsilon$. $\iff$ ${}^\forall x\in X, {}^\forall\varepsilon>0, {}^\exists\delta>0; U_\delta(x)\subset f^{-1}(U_\varepsilon(f(x)))$

· *The above def. is equivalent to the def. in top.*

($\Rightarrow$) ${}^\forall V\in\mathcal{O}_Y$, ${}^\forall x\in f^{-1}(V)$, $f(x)\in V$ and by V being open, ${}^\exists\varepsilon>0; U\varepsilon(f(x))\subset V$. By the assumption, ${}^\exists\delta>0; U_\delta(x)\subset f^{-1}(U_\varepsilon(f(x)))\subset f^{-1}(V)$. This implies $f^{-1}(V)$ is open.

($\Leftarrow$) ${}^\forall x\in X, {}^\forall\varepsilon>0$, by the assumption, ${}^\forall x'\in V:=f^{-1}(U_\varepsilon(f(x))), {}^\exists\delta>0; U_\delta(x')\subset V$. It can be taken as $x'=x$, and hence, $U_\delta(x)\subset V$. ∎

For a map. $f:X\to(Y,\mathcal{O}_Y)$, $\mathcal{O}_X=f^{-1}(\mathcal{O}_Y)$: the **weakest top.** of X such that f is conti.

For a map. $f:(X,\mathcal{O}_X)\to Y$, $\mathcal{O}_Y=\{B\subset Y; f^{-1}(B)\in\mathcal{O}_X\}$: the **strongest top.** of Y such that f is conti.

$(X,\mathcal{O}_X)$: a top. sp., $\sim$: a equivalent relation in X. For the quotient set $Y=X/\sim\ni[x]\ni y \overset{\text{def}}{\iff} x\sim y$, $f:X\to X/\sim; f(x)=[x]$, a **quotient top.** $\mathcal{O}_{X/\sim}\ni B\subset X/\sim \overset{\text{def}}{\iff} f^{-1}(B)\in\mathcal{O}_X$, where a **equivalence relation** $\sim$ is a binary relation that is reflexive, symmetric and transitive; $x\sim x$, $x\sim y\Rightarrow y\sim x$, $x\sim y, y\sim z\Rightarrow x\sim z$.

The above top's are called **induced top's**.

For a map. $f:(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$, f is an **open map.** if $f(\mathcal{O}_X)\subset\mathcal{O}_Y$, and f is a **closed map.** if $f(\mathcal{C}_X)\subset\mathcal{C}_Y$.

$(X,\mathcal{O}_X)$ is **homeomorphic** to $(Y,\mathcal{O}_Y)$, or X,Y are **homeomorphic** if ${}^\exists f:X\to Y$ is 1-1 onto, conti. and an open map.

For $(X_1,\mathcal{O}_1),\ldots,(X_n,\mathcal{O}_n)$, a **product top.** of $X=\prod_{k=1}^n X_k$: $\mathcal{O}_X\ni\bigcup_\lambda U_\lambda: U_\lambda\in\prod_{k=1}^n\mathcal{O}_k$.

In infinite case, the product top. is the weakest top. such that every projections are continuous. That is, let $(X_\lambda,\mathcal{O}_\lambda)_{\lambda\in\Lambda}$: top. sp. $X=\prod_{\lambda\in\Lambda}X_\lambda$ and $P_\lambda:X\to X_\lambda; (x_\lambda)\mapsto x_\lambda$: a **projection**, The product top. $\mathcal{O}_X$ is a top. such that $\{P_\lambda^{-1}U_\lambda, U_\lambda\in\mathcal{O}_\lambda\}$ is a quasi basis, i.e., a family of **cylinder sets** $\{\bigcap_{k=1}^n P_{\lambda_k}^{-1}U_{\lambda_k}; U_{\lambda_k}\in\mathcal{O}_{\lambda_k}, k=1,2,\ldots,n, n\geq 0\}$ is a top. basis. (The intersection of 0-numbers is the total set X.) More concretely,
$\mathcal{O}_X\ni V=\bigcup_{\alpha\in A}V_\alpha; V_\alpha=\bigcap_{k=1}^{n_\alpha}P_{\lambda_{\alpha,k}}^{-1}U_{\lambda_{\alpha,k}}; (U_{\lambda_{\alpha,k}}\in\mathcal{O}_{\lambda_{\alpha,k}}, k=1,\ldots,n_\alpha, n_\alpha\geq 0)$,

2 Topological structures; Compacts, Connected and Separation axioms

2.1 Compacts=cpt's

A set is **compact=cpt** if for an arbitrary open covering, there exists a finite open covering, where an **open covering=O.C.** is a family of open subsets such that the union contains the set.

$K\subset X$: **cpt** $\overset{\text{def}}{\iff}$ ${}^\forall\mathcal{U}\subset\mathcal{O}; K\subset\bigcup\mathcal{U}, {}^\exists U_1,\ldots,U_n\in\mathcal{U}; K\subset\bigcup_{k=1}^n U_k$.

Note that ${}^\forall\mathcal{U}\subset\mathcal{O}; K\subset\bigcup\mathcal{U}, {}^\exists U_1,\ldots,U_n\in\mathcal{U}; K\subset\bigcup_{k=1}^n U_k \iff {}^\forall\mathcal{F}\subset\mathcal{C}; \bigcap\mathcal{F}\cap K=\emptyset, {}^\exists F_1,\ldots,F_n\in\mathcal{F}; \bigcap_{k=1}^n F_k\cap K=\emptyset \iff {}^\forall\mathcal{F}\subset\mathcal{C}; {}^\forall n\geq 1, {}^\forall F_1,\ldots,F_n\in\mathcal{F}, \bigcap_{k=1}^n F_k\cap K\neq\emptyset$, i.e., $\mathcal{F}$ has a **finite intersection property** in K, then $\bigcap\mathcal{F}\cap K\neq\emptyset$

Hence, $K\subset X$: cpt $\iff$ ${}^\forall\mathcal{F}\subset\mathcal{C}$ having finite intersection propery in K, i.e., ${}^\forall n\geq 1, {}^\forall F_1,\ldots,F_n\in\mathcal{F}, \bigcap_{k=1}^n F_k\cap K\neq\emptyset, \bigcap\mathcal{F}\cap K\neq\emptyset$
$\iff$ ${}^\forall\mathcal{E}\subset 2^X; {}^\forall n\geq 1, {}^\forall E_1,\ldots,E_n\in\mathcal{E}, \bigcap_{k=1}^n\overline{E_k}\cap K\neq\emptyset, \bigcap\overline{\mathcal{E}}\cap K\neq\emptyset$, where $\bigcap\overline{\mathcal{E}}=\bigcap\{\overline{E}; E\in\mathcal{E}\}$. .

If a total set is cpt, then it is called a **cpt (top.) sp.**, if a subset is cpt, then it is called a **cpt (sub)set**, and if a closure is cpt, then it is called a **relatively cpt**.

· *A closed subset F of a cpt set C is cpt.*

${}^\forall\mathcal{U}\subset\mathcal{O}$: O.C. of F, $\mathcal{U}\cup\{F^c\}$ is an O.C. of C. ${}^\exists U_1,\ldots,U_n\in\mathcal{U}; C\subset\bigcup U_k\cup F^c$. Hence $F\subset\bigcup U_k$. ∎

· *A cpt subset C of a Hausdorff sp. is closed.*

${}^\forall x \in C^c$: fixed. ${}^\forall y \in C, {}^\exists U_y, V_y \in \mathcal{O}; x \in U_y, y \in V_y, U_y \cap V_y = \emptyset$. $\{V_y\}_{y\in C}$ is an O.C. of C. ${}^\exists y_1, \ldots, y_n \in C; C \subset \bigcup V_{y_k}$. Hence, $U := \bigcap U_{y_k} \in \mathcal{O}$ and $x \in U \subset C^c$. Therefore C^c is open, i.e., C is closed. ■

· *A continuous image of a cpt set is also cpt.*

· *A continuous map. from a cpt top sp. to a Hausdorff sp. is an open map. Especially, if it is 1-1 onto, then it is homeomorphic.*

A closed set of a cpt set is cpt and its conti. image is also cpt. Moreover, the cpt set of a Hausdorff sp. is closed. ■

· **Tychonoff's theorem**: *A product top. sp. of any numbers of cpt sp's is cpt and vice verse.*

${}^\forall \lambda \in \Lambda, X_\lambda$: *cpt* $\iff X = \prod_{\lambda\in\Lambda} X_\lambda$: *cpt*

($\Leftarrow$) Projections $P_\lambda : X \to X_\lambda; (x_\lambda) \mapsto x_\lambda$ are conti. and a cpt image $X_\lambda = P_\lambda X$ is also cpt.

($\Rightarrow$) For a fixed arbitrary $\mathcal{F} \subset \mathcal{C}$ having finite intersection property, we show $\bigcap \mathcal{F} \neq \emptyset$.

$$\mathbf{E} = \{\mathcal{E} \subset 2^X; \mathcal{E} \supset \mathcal{F} \text{ having finite intersection property}\}.$$

This is an inductive ordered set by containment relationship as an order. In fact, for a total order part, a family of subsets $\mathcal{F}_0 = \mathcal{E}$ which are all elements of the part is a maximal element of the part. (see (i) in the next question). In this case, by the local maximum property of $\mathcal{E}$, the following hold:

(1) $E_1, \ldots, E_n \in \mathcal{E} \Rightarrow E_1 \cap \cdots \cap E_n \in \mathcal{E}$ (by finite intersection property and loc. max. property).

(2) $A \subset X, {}^\forall E \in \mathcal{E}, A \cap E \neq \emptyset \Rightarrow A \in \mathcal{E}$. (by (1))

For each $\lambda \in \Lambda$, by using a projection P_λ, let

$$\overline{\mathcal{E}}_\lambda := \overline{P_\lambda \mathcal{E}} = \{\overline{P_\lambda E}; E \in \mathcal{E}\}.$$

For $E_1, \ldots, E_n \in \mathcal{E}$, by $\bigcap P_\lambda E_k \supset P_\lambda(\bigcap E_k)$ they have finite intersection property, and by X_λ being cpt, ${}^\exists x_\lambda \in \bigcap \overline{\mathcal{E}}_\lambda$. Hence, it is enough to show $x := (x_\lambda)_{\lambda\in\Lambda} \in \bigcap \overline{\mathcal{E}}$, because $\mathcal{F} \subset \mathcal{E}$ implies $\bigcap \overline{\mathcal{E}} \subset \bigcap \mathcal{F}$ and the proof is finished (For the existence of x, we use axiom of choice), where $\overline{\mathcal{E}} := \{\overline{E}; E \in \mathcal{E}\}$. The proof needs the loc. max. property and finite intersection property. (In general, the existence of an element of the intersection of projections does not ensure the existence an element of the intersection in the product sp. (see (ii) in the next question). ${}^\forall E \in \mathcal{E}$. For a nbd of x; $U := \bigcap_{k=1}^n P_{\lambda_k}^{-1}(U_{\lambda_k}) = P_{\lambda_k}^{-1}(\bigcap_{k=1}^n U_{\lambda_k})$ ($U_{\lambda_k} \in \mathcal{N}(x_{\lambda_k})$: a nbd of x_{λ_k}), $U \cap E \neq \emptyset$ holds. In fact, by $x_\lambda \in \overline{P_\lambda E}$, ${}^\forall U_\lambda \in \mathcal{N}(x_\lambda), U_\lambda \cap P_\lambda E \neq \emptyset$. Hence $P_\lambda^{-1}(U_\lambda) \cap E \neq \emptyset$ (see (iii) in the next question), and since $E \in \mathcal{E}$ is arbitrary and by (2), $P_\lambda^{-1}(U_\lambda) \in \mathcal{E}$ holds. Moreover, by the property (1), $U = \bigcap_{k=1}^n P_{\lambda_k}^{-1}(U_{\lambda_k}) \in \mathcal{E}$ holds. By finite intersection property of $\mathcal{E}$, $U \cap E \neq \emptyset$. Here, note that a family of all nbds of U is a nbd basis of the product sp. X, we have $x \in \overline{E}$. Since $E \in \mathcal{E}$ is arbitrary, we have $x \in \bigcap \overline{\mathcal{E}}$.

Question In the above proof, show the following:

(i) Let $\mathbf{L} \subset \mathbf{E}$ be a total order part of $\mathbf{E}$ and let $\mathcal{F}_0 := \bigcup \mathbf{L} = \{E \in \mathcal{E}; \mathcal{E} \in \mathbf{L}\}$. Then $\mathcal{F}_0 \in \mathbf{E}$, that is, it contains $\mathcal{F}$ and has finite intersection property.

(ii) Make an example of three subsets of $\mathbf{R}^2$ such that the intersection of three is empty, however the intersection of three projections is not empty.

($\to$ each side of an equilateral triangle.)

(iii) show $U_\lambda \cap P_\lambda E \neq \emptyset$. implies $P_\lambda^{-1}(U_\lambda) \cap E \neq \emptyset$. ■

· *In Euclidean sp., cpt=bounded closed.*

($\Rightarrow$) C: cpt. ${}^\exists x_1, \ldots, x_n \in C; C \subset U_1(x_1) \cup \cdots \cup U_1(x_n)$. Hence C: bdd. A cpt set in a Hausdorff sp. is closed. ($\Leftarrow$) C: bdd closed in $\mathbf{R}^n$. ${}^\exists R := \prod_{k=1}^n [a_k, b_k] \supset C$. Assume ${}^\exists \mathcal{U}$: O.C. of C; it has no finite O.C., Divide R equally among 2^n. There exists at least one part R_1 such that it cannot be covered by finite numbers of O.C. and divide it again and we can define R_2. Contnuing these, we have R_k; it cannot be covered by finite number of $\mathcal{U}$. Then ${}^\exists x \in C; R_k \downarrow \{x\}$, ${}^\exists U \in \mathcal{U}; x \in U$ and ${}^\exists U_\delta(x) \subset U$. This implies ${}^\exists K \geq 1; {}^\forall k \geq K, R_k \subset U_\delta(x) \subset U$. However, this contradicts. to the definition of R_k. ■

· *X: cpt, $f : X \to \mathbf{R}$ conti.* $\Rightarrow {}^\exists \max f, \min f$.

$f(X)$: cpt=bdd closed in $\mathbf{R}$. Hence $\sup f(X) = \max f, \inf f(X) = \min f$. In fact,$^\exists y_n \in f(X); y_n \uparrow y_0 := \sup f(X)$. Since $f(X)$ is closed, $y_0 \in f(X)$, i.e., $^\exists x_0 \in X; y_0 = f(x_0) = \max f$. ■

$K \subset X$: **sequentially cpt** $\overset{\text{def}}{\iff}$ $^\forall\{x_n\} \subset K, ^\exists\{n_k\}; x_{n_k} \to {}^\exists x \in K$.

Note that $x_n \to x$ in $(X, \mathcal{O})$ $\overset{\text{def}}{\iff}$ $^\forall U \in \mathcal{N}(x), ^\exists N \geq 1; ^\forall n \geq N, x_n \in U$.

By this it holds that

· $C \in \mathcal{C} \Rightarrow {}^\forall\{x_n\} \subset C; x_n \to {}^\exists x \in X, x \in C$. *Moreover, if X satifies 1st axiom of countability, i.e., existece of a countabile nbd basis, then the inverse holds.*

($\Rightarrow$) If $x \notin C$, then $^\exists U \in \mathcal{N}(x); x \in U \subset C^c$. By $x_n \to x$, $^\exists N \geq 1; ^\forall n \geq N, x_n \in U$. However this contradicts to $x_n \in C$ ($^\forall n \geq 1$).

($\Leftarrow$) Assume existece of countabile nbd basis.

If C is not closed, then C^c is not open. Hence, $^\exists x \in C^c; ^\forall U \in \mathcal{N}(x), U \not\subset C^c$. Let $\mathcal{N}_0(x) = \{U_n\}$ be a countable nbd basis of x. We may assume $U_n \downarrow$. Hence $^\forall n \geq 1, ^\exists x_n \in C \cap U_n$. That is, $x_n \to x$. However $x \notin C$. This cotradicts the assumption. ■

$C \subset X$: **countably cpt**: $\overset{\text{def}}{\iff}$ $^\forall\{U_n\}$: O.C. of C, $^\exists\{n_k\}_{k \leq K}; \{U_{n_k}\}_{k \leq K}$: O.C. of C.

X: **locally cpt**: $\overset{\text{def}}{\iff}$ $^\forall x \in X, ^\exists U \in \mathcal{N}(x); \overline{U}$: cpt.

· *cpt $\Rightarrow$ countably cpt.*

· *seq. cpt $\Rightarrow$ countably cpt.*

It is enough to show that $\bigcap F_n \neq \emptyset$ for $^\forall\{F_n\} \subset \mathcal{C}$; having finite intersection property. For $^\forall N \geq 1$, fix $x_N \in \bigcap_{n=1}^N F_n =: C_N \in \mathcal{C}$. Then $^\exists\{n_k\}; ^\exists x \in X; x_{n_k} \to x$. For $^\forall N \geq 1$, if $n \geq N$, then $x_n \in C_N$. Hence, $x \in C_N$ and $x \in \bigcap_{N \geq 1} C_N = \bigcap_{N \geq 1} F_N$. ■

· *Under 1st axiom of countablity, countably cpt $\Rightarrow$ seq. cpt., that is, they are equivalent.*

$^\forall\{x_n\}$, $F_n := \overline{\{x_k; k \geq n\}}, n \geq 1$ have finite intersection property. Hence, by the assumption, $^\exists x \in \bigcap F_n$. If we assume $^\exists U \in \mathcal{N}(x), ^\exists N \geq 1; ^\forall n \geq N, x_n \notin U$, then $F_N \cap U = \emptyset$, i.e., $F_N \subset U^c$, and hence, $x \in \bigcap_{n \geq 1} F_n \subset U^c$. This contradicts $x \in U$. Therefore, $^\forall U \in \mathcal{N}(x), ^\forall k \geq 1, ^\exists n_k \geq k; x_{n_k} \in U$. Let $U_k, k \geq 1$ be a countable nbd basis of x. If we set $V_K = \bigcup_{k \leq K} U_k$, then it is also a nbd basis. From the above result, $^\forall k \geq 1$ we can take $n_k (\geq k), \upuparrows; x_{n_k} \in V_k$ as follows: First we take $n_1 \geq 1$ and if $n_k > n_{k-1}$ is determined, then we take $n_{k+1} \geq k \vee (n_k + 1)$. ■

Moreover,

· *under 2nd axiom of countability, countabily cpt $\Rightarrow$ cpt, that is, they are equivalent.*

Let $\mathcal{B} \subset \mathcal{O}$ be a countable top. basis. $^\forall U \in \mathcal{O}, ^\exists \mathcal{B}' \subset \mathcal{B}; U = \bigcup \mathcal{B}'$.

For anu O.C. $\mathcal{U}, ^\exists \mathcal{B}' \subset \mathcal{B}; \bigcup_{\mathcal{U}} = \bigcup \mathcal{B}'$. Therefore $^\exists V_1, \ldots, V_n \in \mathcal{B}'$: O.C. For each $k = 1, \ldots, n$, $^\exists U_k \in \mathcal{U}; V_k \subset U_k$. Thus, $\{U_k\}$ is also an O.C. ■

· *Countably cpt + Lindelöf property $\iff$ cpt, where* **Lindelöf property***: For an arbitrary O.C., there exists a countable O.C.*

2.2 Connected

$A \subset X$: **connected** $\overset{\text{def}}{\iff}$ $^\forall U, V \in \mathcal{O}; [U \cap A, V \cap A \neq \emptyset, U \cap V = \emptyset], A \not\subset U \cup V$ $\iff$ If $B \subset A$: open and closed in A, then $B = \emptyset$ or $B = A$, where B: open (closed) in A $\overset{\text{def}}{\iff}$ $^\exists U \in \mathcal{O}(\in \mathcal{C}); B = A \cap U$.

A is **disconnected** = not connected $\overset{\text{def}}{\iff}$ $^\exists U, V \in \mathcal{O}; U \cap A, V \cap A \neq \emptyset, U \cap V = \emptyset, A \subset U \cup V$.

· *$f : X \to Y$: conti. X is connected $\Rightarrow$ $f(X)$ is connected.*

If $f(X)$ is not connected, then $^\exists U, V \in \mathcal{O}_Y; f(X) \cap U, f(X) \cap V \neq \emptyset, f(X) \subset U \cup V$. However this implies X is not connected; $X = f^{-1}(U) \cup f^{-1}(V)$. ■

· *A: connected $\Rightarrow$ $A \subset {}^\forall B \subset \overline{A}$: connected.*

If $A \subset {}^\exists B \subset \overline{A}$ is not connected, then $^\exists U, V \in \mathcal{O}; U \cap B, V \cap B \neq \emptyset, U \cap V = \emptyset, B \subset U \cup V$. However this implies A is not connected. Because if $A \cap U = \emptyset$, then $\overline{A \cap U} = \emptyset$. However $\emptyset \neq B \cap U \subset \overline{A \cap U}$. This contradicts. ■

· *$A_\lambda, \lambda \in \Lambda$: conected $A_\lambda \cap A_{\lambda'} \neq \emptyset$ if $\lambda \neq \lambda'$ $\Rightarrow$ $\bigcup A_\lambda$: connected.*

It is easy by reductio ad absurdum. In fact, if $A := \bigcup A_\lambda$ is not connected, then ${}^\exists U, V \in \mathcal{O}; A \cap U, A \cap V \neq \emptyset, A \subset U \cup V$. For ${}^\forall \lambda$, since A_λ is connected, we have $A_\lambda \subset U$ or $\subset V$. Moreover we have ${}^\exists A_\lambda \subset U, {}^\exists A_{\lambda'} \subset V$. This cotradicts $A_\lambda \cap A_{\lambda'} \neq \emptyset$. ■

· $\prod_{\lambda\in\Lambda} A_\lambda$ *connected* $\iff$ ${}^\forall \lambda, A_\lambda$*: connected.*

($\Rightarrow$) Every projection P_λ is conti. and a conti. image of a connected set is also connected.

($\Leftarrow$) 2 numbers case: $A \times B \ni (a_0, b_0)$: fixed. $\{a_0\} \times B, A \times \{b_0\}$ are connected in the product top. and the intersection contains (a_0, b_0). Hence by the above result, $\{(\{a_0\} \times B) \cup (A \times \{b_0\})\}$ is conneected. Therefore $A \times B$ is connected by the following

$$A \times B = \bigcup_{a\in A} \{a\} \times B = \bigcup_{a\in A} \{(\{a\} \times B) \cup (A \times \{b_0\})\}.$$

In general case: Fix a point $(a_\lambda) \in A := \prod A_\lambda$ (by using the axiom of choice). Let

$$\mathcal{B} = \left\{ \prod B_\lambda; {}^\forall n \geq 1, {}^\forall \lambda_1, \ldots, \lambda_n \in \Lambda, B_{\lambda_k} = A_{\lambda_k}, \text{ and } B_\lambda = \{a_\lambda\} \text{ if } \lambda \neq \lambda_k, \text{ k=1,..., n} \right\}.$$

Since the every element of $\mathcal{B}$ is connected and (a_λ) is an intersection element, $\prod A_\lambda = \bigcup \mathcal{B}$ is connected. ■

A **connected component** is a maximal conected set.

· *Connected sets of* $\mathbf{R}$ *are only intervals* $[a, b], (a, b], (a, b), (a, b]$*, where if* $a = b$*, then* $[a, a] = \{a\}$ *and the others are empty, or if* $a = -\infty$*, then* $[a, (a = (-\infty$*, or if* $b = +\infty$*, then* $b], b) = +\infty)$*.*

· *A real-values conti. function on a connected set satisfies the* **intermediate value theorem***.*

X*: connected,* $f : X \to \mathbf{R}$*: conti.* $\Rightarrow {}^\forall a, b \in X; f(a) < f(b), {}^\forall \gamma \in (f(a), f(b)), {}^\exists c \in X; f(c) = \gamma$*.*

The conti. image of connected is connected and it is an interval in $\mathbf{R}$. Hence the claim is clear. ■

X: **path-connected** $\overset{\text{def}}{\iff}$ ${}^\forall x, x' \in X, {}^\exists x \to x'$: a path, i.e., ${}^\exists f : [0, 1] \to X$: conti., $f(0) = x, f(1) = x'$, $f([0, 1])$ is called a **path from** x **to** x' and denoted as $x \to x'$.

· *path-connected* $\Rightarrow$ *connected.*

In $\mathbf{R}^2$ *the union* X *of the following line segments is connected, but not path-connected:*

$(0, 1] \times \{0\}, \{0\} \times (0, 1], \{1/n\} \times [0, 1] (n = 1, 2, ...)$

It is clear that $X \ni \{(0, 0)\}$ is not path-connected.

The union X_1 of $\{0\}\times(0, 1], \{1/n\}\times[0, 1] (n = 1, 2, ...)$ is path-connected, i.e., connected and $\overline{X}_1 \backslash \{O\} = X$ is also connected. ■

X: **locally connected** $\overset{\text{def}}{\iff}$ ${}^\forall x \in X, {}^\exists \mathcal{B}(x)$: a nbd-basis; ${}^\forall V \in \mathcal{B}(x)$ is connected.

· X*: loc. connected* $\iff$ ${}^\forall U \in \mathcal{O}, {}^\forall C \subset U$*: connected comp. is open.*

($\Rightarrow$) ${}^\forall U \in \mathcal{O}, {}^\forall C \subset U; C \neq \emptyset$. ${}^\forall x \in C$, by loc.connected, ${}^\exists V \in \mathcal{N}(x); V \subset C, V$: connected. In fact, if $V \not\subset C$, then $V \cup C$ is conneceted. This contradicts maximal property of C. Therefore, C is open.

($\Leftarrow$) We show $\{V$; connected open nbd's $\}$ is a top. basis. Since $\{x\}$ is connected, ${}^\forall U \in \mathcal{O}, {}^\forall x \in U, x \in {}^\exists V \subset U; V$: connected comp. By the assumption, V is open, and hence, the family of all such V's is a top. basis. ■

2.3 Homeomorphic, compact and connected

· $f : X \to Y$*: homeo.* X *cpt (or connected)* $\Rightarrow f(X)$*: cpt (or connected).*

· $\mathbf{R} \not\approx \mathbf{R}^2$ *(not homeo.)* $\quad$ $\mathbf{R} \setminus \{0\}$ is not connected. However, $\mathbf{R}^2 \setminus \{(a, b)\}$ is connected. ■

· $\mathbf{R} \approx (0, 1) \not\approx (0, 1]$. $\quad$ $(0, 1] \setminus \{1\} = (0, 1)$ is connected, however $(0, 1) \setminus \{a\}$ is not connected. ■

· $\mathbf{R} \not\approx \mathbf{S}^1$. $\quad$ $\mathbf{S}^n$: cpt, $\mathbf{R}^n$: not cpt. ■

2.4 Separation axioms

Recall $\mathcal{N}_{\mathcal{O}}(x) := \mathcal{N}(x) \cap \mathcal{O}$: open nbds of x, $\mathcal{N}_{\mathcal{C}}(x) := \mathcal{N}(x) \cap \mathcal{C}$: closed nbds of x.

T_1: **(Fréchet's axiom)** ${}^\forall x, y; x \neq y, {}^\exists U_x \in \mathcal{N}_{\mathcal{O}}(x); y \notin U_x \iff {}^\forall x, \bigcap \mathcal{N}(x) = \{x\} \iff {}^\forall x, \{x\} \in \mathcal{C}$.

T_2: **(T_2 sp.=Hausdorff sp.)** ${}^\forall x, y; x \neq y, {}^\exists U_x \in \mathcal{N}_{\mathcal{O}}(x), V_y \in \mathcal{N}_{\mathcal{O}}(y); U_x \cap V_y = \emptyset. \iff {}^\forall x, \{x\} = \bigcap(\mathcal{N}(x) \cap \mathcal{C})$.

($\Rightarrow$) If ${}^\exists y \neq x; y \in \bigcap \mathcal{N}_{\mathcal{C}}(x)$, then ${}^\forall F_x \in \mathcal{N}_{\mathcal{C}}(x), y \in F_x$. This cotradicts T_2. In fact, by T_2, ${}^\exists U_x \in \mathcal{N}_{\mathcal{O}}(x); \overline{U_x} \not\ni y$. $F_x := \overline{U_x}$ does not satisfies the above result.

($\Leftarrow$) ${}^\forall x, y; x \neq y$, by the assumption, ${}^\exists F_x \in \mathcal{N}_{\mathcal{C}}(x); y \notin F_x$. Hence, $U_x := F_x^o \in \mathcal{N}_{\mathcal{O}}(x), V_y := F_x^c \in \cap N_{\mathcal{O}}(y)$. ■

· *In this sp. uniqueness of a limit of a sequence holds.*

In $\mathbf{N}$, if we define $\mathcal{C} \ni \emptyset, \mathbf{N}$ and all finite sets, then this is a T_1-sp., however not T_2. Moreover, $\{n\}$ converges to any $k \in \mathbf{N}$. ■

T_3: **(Vietoris's axiom)** ${}^\forall x, {}^\forall F \neq \emptyset; x \notin F \in \mathcal{C}, {}^\exists U_x, V_F \in \mathcal{O}; x \in U_x, F \subset V_F, U_x \cap V_F = \emptyset$.

T_4: **(Tietoze's axiom)** ${}^\forall F, F' \neq \emptyset, \in \mathcal{C}; F \cap F' = \emptyset, {}^\exists U_F, U_{F'} \in \mathcal{O}; F \subset U_F, F' \subset U_{F'}, U_F \cap U_{F'} = \emptyset$.

Clearly, · $T_2 \Rightarrow T_1$.

A **regular sp.** $= T_1 + T_3 \iff T_2 + T_3$

A **normal sp.** $= T_1 + T_4 \iff T_2 + T_4$

· *normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff $\Rightarrow$ T_1*

T^*: ${}^\forall x, {}^\forall F \in \mathcal{C}; x \notin F, {}^\exists f : X \to [0, 1]$; conti. $f(x) = 0$, $f = 1$ on F.

A **completely regular sp.** $= T_1 + T^*$

· *normal $\Rightarrow$ completely regular $\Rightarrow$ regular*

1st $\Rightarrow$ is clear from the following lemma and that singleton is closed by T_1. 2nd $\Rightarrow$ is clear by $U_x = \{f < 1/2\}, V_F = \{f > 1/2\}$, where f is a conti, ft in T^*. ■

· **[Urysohn's Lemma]**

$T_4 \iff {}^\forall F, F' \neq \emptyset, \in \mathcal{C}; F \cap F' = \emptyset, {}^\exists f : X \to [0, 1]$*; conti.* $f = 0$ *on* F, $f = 1$ *on* F'.

First,

· $T_4 \iff T_4'$*:* $\emptyset \neq {}^\forall F$*: closed,* ${}^\forall U \supset F$*: open,* ${}^\exists V \in \mathcal{O}; F \subset V \subset \overline{V} \subset U$.

For $F, F' \neq \emptyset, \in \mathcal{C}; F \cap F' = \emptyset$. let $U := (F')^c \supset F$. By T_4' we can make a conti. ft f as follows: Let $G_1 = U\ (F')^c$. ${}^\exists G_0 \in \mathcal{O}; F \subset G_0 \subset \overline{G_0} \subset G_1$. ${}^\exists G_{1/2} \in \mathcal{O}; \overline{G_0} \subset G_{1/2} \subset \overline{G_{1/2}} \subset G_1$. Moreover, ${}^\exists G_{1/4}, G_{3/4} \in \mathcal{O}$ between $G_0, G_{1/2}$ or $G_{1/2}, G_1$ and we take $G_r \in \mathcal{O}$ for $r = k/2^n$. Define $f = 0$ on G_0, $= 1$ on G_1^c and $f(x) = \inf\{r; x \in G_r\}$ for $x \in G_1 \setminus G_0$. Then, f is conti.

For the inverse, $U_F = \{f < 1/2\}, U_{F'} = \{f > 1/2\}$. ■

· *cpt Hausorff is normal.*

By using that a closed set of a cpt set is cpt and T_2, we can show T_3, and moreover, T_4. ■

· *Regular+2nd axiom of countability $\Rightarrow$ normal.*

${}^\forall F_1, F_2 \neq \emptyset, \in \mathcal{C}; F_1 \cap F_2 = \emptyset$. $\mathcal{B}$: a countable top. basis. Define $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}$ as follow:

$\mathcal{B}_1 \equiv \{B_{m_j}\} \ni B \iff x \in F_1, {}^\exists B \in \mathcal{B}; x \in B \subset \overline{B} \subset F_2^c$

$\mathcal{B}_2 \equiv \{B_{n_k}\} \ni B \iff y \in F_2, {}^\exists B \in \mathcal{B}; y \in B \subset \overline{B} \subset F_1^c$

and set $U_1 = B_{m_1}, V_1 = B_{n_1} \setminus \overline{U_1}$, $U_k = B_{m_k} \setminus (\bigcup_{j=1}^{k-1} V_j), V_k = B_{n_k} \setminus (\bigcup_{j=1}^{k} U_j)$. Then $U_j \cap V_k = \emptyset$ and the following $U, V; U \cap V = \emptyset$.

$$U := \bigcup_{k \geq 1} U_k = \bigcup_{k \geq 1} B_{m_k} \supset F_1, V := \bigcup_{k \geq 1} V_k = \bigcup_{k \geq 1} B_{n_k} \supset F_2.$$

■

3 In metric sp's

A metric sp. has a countable nbd basis; $U_{1/n}(x)$, and satisfies 1st axiom of countability. Moreover, it satisfies T_1 and T_4, and hence, it is normal.

· *In a metric sp. separable (existence of a countable dense set) $\iff$ 2nd axiom of countability (existence of countable top. basis).*

· *totally bounded, i.e.,* $^\forall \varepsilon > 0, ^\exists U_1, \ldots, U_n; 0 < D(U_k) \le \varepsilon, X \subset \bigcup U_k \Rightarrow$ *separable, where* $D(U) = \sup_{x,y \in U} d(x,y)$.

· *In a metric sp. the following are equivalent: cpt, countable cpt, seq. cpt, totally bdd and complete, where* **complete** $\overset{\text{def}}{\iff}$ *An arbitrary Cauchy seq. converges, and a* **Cauchy seq.** $\{a_n\} \overset{\text{def}}{\iff} d(a_n, a_m) \to 0 \ (m, n \to \infty)$.

· *Euclidean sp's are complete.*

It is enough to show 1*-dim. case. A Cauchy seq. is bdd. By Bolzano-Weierstrass,* $^\exists$ *converging sub-seq. Hence, (it is easy to show that) an original Cauchy seq. converges to the same limit.* ■

Equivalent metric: *In a metric sp.* (X, d)*, another metric* d' *is equivalent to* d*, i.e.,* $d' \cong d$*; if the top. under* d' *is the same as the one under* d*.*

· $d' \cong d \iff d(x_n, x) \to 0 \iff d'(x_n, x) \to 0$

$d_1(x,y) := d(x,y)/(1 + d(x,y)), d_2(x,y) = d(x,y) \wedge 1 \Rightarrow d_1 \cong d_2 \cong d$.

Product metric sp's

(X_n, d_n)*: metric sp.*

Finite numbers of n*: Define a metric of* $X = \prod_{n=1}^N X_n$ *as* $d(x,y) = (\sum_{n=1}^N d_n(x_n, y_n)^2)^{1/2}$.

Infinitely many numbers: $X = \prod_{n=1}^\infty X_n$ *has a metric:*

$$d(x,y) := \sum_{n=1}^{\infty} d_n(x_n, y_n) \wedge 2^{-n}.$$

A set metric $d(A, B) := \inf\{d(x,y); x \in A, y \in B\}$

Especially, $d(x, A) := \inf_{y \in A} d(x,y)$.

· *It holds* $|d(x,A) - d(y,A)| \le d(x,y)$*, and hence,* $d(x,A)$ *is conti. in* x.

$^\forall z \in A, d(x,A) \le d(x,z) \le d(x,y) + d(y,z)$*. Since* $z \in A$ *is arbitrary, it holds* $d(x,A) \le d(x,y) + d(y,A)$*, i.e.,* $d(x,A) - d(y,A) \le d(x,y)$*. By exchanging* x, y*, we have desired result.* ■